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ALGORITHMIC PROBLEMS OF AUTOMATA *

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In this paper, we shall introduce the concept "exteded automata" and reduce the decision problem whether a finite semigroup is an amalgamation base for all semigroups or not to an algorithmic problem of automata.

1 Automata

Definition. A finite automaton $\mathcal{A}=(\Sigma, X, E, I, T)$: Σ is a finite set of states, I $(\subseteq \Sigma)$ is a set of initial states, T $(\subseteq \Sigma)$ is a set of terminal states, X is a set of finite letters, E is a subset of the product set $\Sigma\times X\times\Sigma$.

Each element of E is an edge of the form (σ, x, τ) and x is the label of the edge.

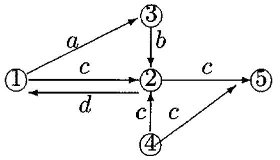
DefinitionA path P on $\mathcal{A}$ is a sequence of edges :
$P=(\sigma_{0}, x_{1}, \sigma_{1}), (\sigma_{1}, x_{2}, \sigma_{2}), \ldots, (\sigma_{t-1}, x_{t}, \sigma_{t})$ (t is a length of the path P)
The signunature $Sg(P)$ of P is a word $x_{1}x_{2}\ldots x_{t}$.
$\sigma_{0}\in I,\ \sigma_{t}\in T\Rightarrow Sg(P)$ is an acceptable word, which assigns a move from σ_{0} to σ_{t}.

Definition. For a finite automaton $\mathcal{A}=(\Sigma, X, E, I, T)$, let $L(\subset X^{*})$ be a set of all acceptable words. Simply, $L=\mathcal{L}(\mathcal{A})$ the language L is the accepted by $\mathcal{L}(\mathcal{A})$.
In general, L is called a regular language.

*This is an absrtact and the paper will appear elsewhere.

Example.

$\mathcal{A}$=$(\{1,2,3,4,5\},\{a,b,c\},E,\{1\},\{5\})$



$\mathcal{L}(\mathcal{A})$=$\{\{abd,cd\}^*cc,abc\}$

Definition. Given a finite automaton $\mathcal{A}=(\Sigma,X,E,I,T)$. For a word $w\in X^*$, if there exist states $\sigma_1,\sigma_2,\sigma_3,\sigma_4$ and paths P_1 from σ_0 to σ_1, P_2 from σ_2 to σ_1, P_3 from σ_2 to σ_3 with $w=Sg(P_1)=Sg(P_2)=Sg(P_3)$, then construct a new path with $Sp(P)=w$ from σ_0 to σ_3. An extended path P with $Sg(P)=w'$ from σ to β is a path obtained by finitely many constructing new paths for subword w of w' in $\mathcal{A}$. i.e., if $w=w'w_1w_2w''$, $\sigma_0\overset{w_1}{\to}\sigma_1\overset{w_2}{\to}\sigma_2\overset{w_2}{\leftarrow}\sigma_3\overset{w_2}{\to}\sigma_4\overset{w_1w_2}{\leftarrow}\sigma_5\overset{w_1}{\to}\sigma_6\overset{w_2}{\to}\sigma_7$ then $\sigma_0\overset{w_1}{\to}\sigma_1\overset{w_2}{\to}\sigma_4\overset{w_1w_2}{\leftarrow}\sigma_5\overset{w_1}{\to}\sigma_6\overset{w_2}{\to}\sigma_7$ and $\sigma_0\overset{w_1w_2}{\to}\sigma_7$

The extended regular language $\mathcal{L}^e(\mathcal{A})$ consists of all signatures of extended path from initial states to terminal states of $\mathcal{A}$.

Question. What a kind of language is an extended regular language of an automaton ?

We give a concrete description of extended automata.

Theorem. Given a finite automaton $\mathcal{A}=(\Sigma,X,E,I,T)$. For $\alpha,\beta\in\Sigma$, let $\mathcal{A}_{\sigma,\beta}$ be an automaton accepting all sigunatures of paths from σ to β.

For $\alpha,\beta\in\Sigma$, let $\mathcal{B}(\mathcal{A})_{\alpha,\beta}$ with a unique initial state α and a unique terminal state β be an automaton accepting all words in $\bigcup\limits_{\gamma\in\Sigma}\mathcal{L}(\mathcal{A}_{\alpha,\gamma})\cap\mathcal{L}(\mathcal{A}_{\beta,\gamma})\cap\mathcal{L}(\mathcal{A}_{\gamma,\beta})$.

$n=1$, $\mathcal{A}^{(1)}$ is an automaton obtained by pasting $\mathcal{B}(\mathcal{A})_{\sigma,\beta}$ to $\mathcal{A}$ by identifying α,β of $\mathcal{A}$ with α,β of $\mathcal{B}_{\alpha,\beta}$ for all $\alpha,\beta\in\Sigma$.

$n=k+1$, $\mathcal{A}^{(n)}$ is an automaton obtained by pasting $\mathcal{B}(\mathcal{A}^{(k)})_{\sigma,\beta}$ to $\mathcal{A}^{(k)}$ by identifying α,β of $\mathcal{A}$ with α,β of $\mathcal{B}_{\alpha,\beta}$ for all $\alpha,\beta\in\Sigma$.with $\mathcal{A}^{(k)}_{\alpha,\beta}$ pasted for all $\alpha,\beta\in\Sigma$.

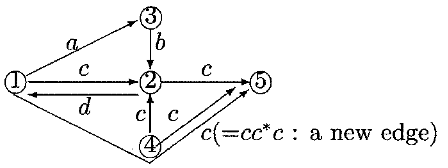
$\mathcal{A}^e=\bigcup\limits_{n=1}^{\infty}\mathcal{A}^{(n)}$. (Possibly, an infinite automaton)

Then

the extended regular language $\mathcal{L}^e(\mathcal{A})=\mathcal{L}(\mathcal{A}^e)$.

Example.

$\mathcal{A}$=$(\{1,2,3,4,5\},\{a,b,c\},E,\{1\},\{5\})$



$\mathcal{L}^e(\mathcal{A}) = \mathcal{L}(\mathcal{A}^e)$=$\{\{abd, cd\}^*cc, abc, \{abd, cd\}^*c\}$

2 Decision problems

Extended automata are applicable to decision problems of semigroup amalgamations.

Result 1[5,Theorem 2.1]. Let S be a finte semigroup and $\mathcal{I}(S^1)$ the injective hull of the left S-set S^1.
Then S has the representation extension property if and only if for any right S-set X_S, the canonical map : $X \to X \otimes_S \mathcal{I}(S^1)$ $(x \longmapsto x \otimes 1)$ is injective.

Result 2[6.the main theorem]. The decision problem whether a finite semigroup has the epresentation extension property or not is decidable.

Lemma. Let X be a right S-set and Y a left S-set. Then $x \otimes y = x' \otimes y'$ *in* $X \otimes Y$ if and only if there exist $s_1, \cdots, s_n$, $t_1, \cdots, t_n \in S^1$, $x_1, \cdots, x_n \in X$ and $y_2, \cdots, y_n \in Y$ such that

$$\begin{array}{rclcrcl} x & = & x_1 s_1, & \quad & s_1 y & = & t_1 y_2 \\ x_1 t_1 & = & x_2 s_2, & & s_2 y_2 & = & t_2 y_3 \\ & \vdots & & & & \vdots & \\ x_{n-1} t_{n-1} & = & x_n s_n, & & s_n y_n & = & t_n y' \\ x_n t_n & = & x' & & & & \end{array} \tag{1}$$

Then we call the system of equations (1) a scheme of length n X and Y joining (x, y) to (x', y').

proposition. Let S be a finite regular semigroup. Then S is left absolutely flat if and only if for a $\mathcal{R}$-class of S, a right S-set X and a left S-set Y, $x \otimes y = x' \otimes y'$ in $(xS \cup x'S) \otimes_S Y$

for all $x, x' \in X$ and all $y, y' \in Y$ such that there exists $s_i, t_i \in S$ and $x_i \in X$, $y_i \in Y$ such that

$$\begin{array}{rclrcl} x & = & x_1 s_1, & s_1 y & = & t_1 y_2 \\ x_1 t_1 & = & x_2 s_2, & s_2 y_2 & = & t_2 y_3 \\ & \vdots & & & \vdots & \\ x_{n-1} t_{n-1} & = & x_n s_n, & s_n y_n & = & t_n y' \\ x_n t_n & = & x' & & & \end{array} \tag{2}$$

and there exists some $1 \leq j \leq n$ such that

$xS \subseteq x_1 t_1 S \subseteq \cdots \subseteq x_{j-1} t_{j-1} S \subseteq x_j t_j S^1 \supseteq x_{j+1} t_{j+1} S \supseteq \cdots \supseteq x_{n-1} t_{n-1} S \supseteq x' S$.

Hereafter we call such a set of equations (2) a upward-convex scheme joining from (x, y) to (x', y') over X and Y.

Problem 1. The decision problem whether a finite semigroup is left absolutely flat or not.

an idea for a positive solution of the problem: There are a correspondence between schemes on tensor products of S-sets and paths from initial states and terminal states on finite automata.

The problem is reduced to decision problem whether or not the language of an automaton $\mathcal{A}$ are contained in the language of the extended automaton of a smaller automaton than $\mathcal{A}$ (by deleting some states!).

Result 3. [5, Theorem 2.2] and [4, Theorem 6.11]. A semigroup S is an amalgamation base for all semigroups if and only if for each a right S-set X, a left S-set Y and a S-biset N which is the injective hull of the S-biset S^1,

the map : $X \otimes Y \longrightarrow X \otimes N \otimes Y$ $(x \otimes y \longrightarrow x \otimes 1 \otimes y)$ is injective.

Problem 2. The decision problem whether a finite semigroup is an amalgamation base for all semigroups or not.

an idea for a positive solution of the problem: There are a correspondence between schemes on tensor products of S-sets and paths from initial states and terminal states on finite automata.

The problem is reduced to decision problem whether or not the language of an automaton $\mathcal{A}$ are contained properly in the language of the extended automaton of a smaller finite automaton than $\mathcal{A}$(by deleting some states!).

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